Recursion

Genome 559: Introduction to Statistical and Computational Genomics

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The merge sort algorithm

1. Split your list into two halves

2. Sort the first half

3. Sort the second half

4. Merge the two sorted halves, maintaining a sorted order
Divide-and-conquer

- The basic idea behind the merge sort algorithm is to divide the original problem into two halves, each being a smaller version of the original problem.

- This approach is known as *divide and conquer*
  - Top-down technique
  - Divide the problem into *independent* smaller problems
  - Solve smaller problems
  - Combine smaller results into a larger result thereby “conquering” the original problem.
Merge sort – the nitty gritty

The merge sort algorithm
1. Split your list into two halves
2. Sort the first half
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That’s simple
Careful bookkeeping, but still simple

If I knew how to sort, I wouldn’t be here in the first place?!!
Merge sort – the nitty gritty

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That’s simple

So …

how are we going to sort the two smaller lists?

Here’s a crazy idea: let’s use merge sort to do this

```
def mergeSort(lst):
    half1 = lst[:len(lst)//2]
    half2 = lst[len(lst)//2:]
    half1_sorted = mergeSort(half1)
    half2_sorted = mergeSort(half2)
    return merge(half1_sorted, half2_sorted)
```
You must be kidding, right?

- WHAT?
- This function has no loop?
- It seems to refer to itself!
- Where is the actual sort?
- What’s going on???

def mergeSort(list):
    half1 ← first half of list
    half2 ← second half of list
    half1_sorted = mergeSort(half1)
    half2_sorted = mergeSort(half2)
    list_sorted = merge(half1_sorted, half2_sorted)
    return list_sorted

this is making me dizzy!
Let’s take a step back ...
Factorial

- A simple function that calculates n!

```python
# This function calculated n!
def factorial(n):
    f = 1
    for i in range(1,n+1):
        f *= i
    return f
```

```python
>>> print factorial(5)
120
>>> print factorial(12)
479001600
```

- This code is based on the standard definition of factorial: \( n! = \prod_{k=1}^{n} k \)
Factorial

- But ... there is an alternative **recursive** definition:

\[
 n! = \begin{cases} 
 1 & \text{if} \quad n = 0 \\
 (n-1)! \times n & \text{if} \quad n > 0 
\end{cases}
\]

- So ... can we write a function that calculates \( n! \) using this approach?

```python
# This function calculated n!
def factorial(n):
    if n==0:
        return 1
    else:
        return n * factorial(n-1)
```

- Well ...
  We can! It works! And it is called a **recursive** function!
Why is it working?

# This function calculated n!
def factorial(n):
    if n==0:
        return 1
    else:
        return n * factorial(n-1)

factorial(5)

5 * factorial(4)

4 * factorial(3)

3 * factorial(2)

2 * factorial(1)

1 * factorial(0)

1
Recursion and recursive functions

- A function that calls itself, is said to be a **recursive** function (and more generally, an algorithm that is defined in terms of itself is said to use recursion or be recursive).

  *(A call to the function “recurs” within the function; hence the term “recursion”)*

- In may real-life problems, recursion provides an intuitive and natural way of thinking about a solution and can often lead to very elegant algorithms.
If a recursive function calls itself in order to solve the problem, isn’t it circular? *(in other words, why doesn’t this result in an infinite loop?)*

Factorial, for example, is not circular because we eventually get to 0!, whose definition does not rely on the definition of another factorial and is simply 1.

- This is called a **base case** for the recursion.
- When the base case is encountered, we get a closed expression that can be directly computed.
Defining a recursion

- Every recursive algorithm must have two key features:
  1. There are one or more **base cases** for which no recursion is applied.
  2. All recursion chains eventually end up at one of the base cases.

*The simplest way for these two conditions to occur is for each recursion to act on a smaller version of the original problem. A very small version of the original problem that can be solved without recursion then becomes the base case.*
This is fun!
Let’s try to solve (or at least think of) other problems using recursion
String reversal

- Divide the string into *first character and all the rest*
- Reverse the “rest” and append the first character to the end of it

```
# This function reverses a string
def reverse(s):
    return reverse(s[1:]) + s[0]
```

`s[1:]` returns all but the first character of the string. We *reverse* this part (`s[1:]`) and then concatenate the first character (`s[0]`) to the *end*. See how simple and elegant it is! No loops!
# This function reverses a string

def reverse(s):
    return reverse(s[1:]) + s[0]

>>> print reverse("hello world")
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
  File "<stdin>", line 2, in reverse
RuntimeError: maximum recursion depth exceeded

What just happened? There are 1000 lines of errors!
String reversal – Duh!

- **Remember**: To build a correct recursive function, we need a **base case** that doesn’t use recursion!

  We forgot to include a base case, so our program is an infinite recursion. Each call to “reverse” contains another call to reverse, so none of them return.

  *Each time a function is called it takes some memory. Python stops it at 1000 calls, the default “maximum recursion depth.”*

- **What should we use for our base case?**
String reversal - Yeah

- Since our algorithm is creating shorter and shorter strings, it will eventually reach a stage when $S$ is of length 1 (one character).
- Since a string of length 1 is its own reverse, we can use it as the base case.

```python
# This function reverses a string
def reverse(s):
    if len(s) == 1:
        return s
    else:
        return reverse(s[1:]) + s[0]

>>> print reverse("hello world")
"dlrow olleh"
```
Search a sorted list
(think phonebook)

- How would you search a sorted list to check whether a certain item appears in the list and where?
  - Random search *(yep, I know, this is a stupid algorithm)*
  - Serial search – $O(n)$
  - Binary search

*hunting a lion in the desert*
Binary search

The binary-search algorithm

1. If your list is of size 0, return “not-found”.
2. Check the item located in the middle of your list.
3. If this item is **equal** to the item you are looking for: you’re done! Return “found”.
4. If this item is **bigger** than the item you are looking for: do a binary-search on the **first half** of the list.
5. If this item is **smaller** than the item you are looking for: do a binary-search on the **second half** of the list.

How long does it take for this algorithm to find the query item (or to determine it is not in the list)?
Towers of Hanoi

- There are three posts and 64 concentric disks shaped like a pyramid.
- The goal is to move the disks from post A to post B, following these three rules:
  1. You can move only one disk at a time.
  2. A disk may not be “set aside”. It may only be stacked on one of the three posts.
  3. A larger disk may never be placed on top of a smaller one.
Towers of Hanoi

*Towers-of-Hanoi algorithm (for an “n disk tower”)*

1. Move an “n-1 disk tower” from source-post to resting-post (use the tower-of-hanoi algorithm)
2. Move 1 disk from source-post to destination-post
3. Move an “n-1 disk tower” from resting-post to destination-post (use the tower-of-hanoi algorithm)

- What should the base case be?

- Assuming each disk move takes 1 second, how long would it take to move a 64 disk tower?
Finally,
let’s get back to our merge sort
The merge sort algorithm

1. Split your list into two halves

2. Sort the first half (*using* merge sort)

3. Sort the second half (*using* merge sort)

4. Merge the two sorted halves, maintaining a sorted order
**The merge sort algorithm**

1. Split your list into two halves
2. Sort the first half *(using merge sort)*
3. Sort the second half *(using merge sort)*
4. Merge the two sorted halves, maintaining a sorted order

---

```python
# Merge two sorted lists
def merge(list1, list2):
    merged_list = []
i1 = 0
i2 = 0

    # Merge
    while i1 < len(list1) and i2 < len(list2):
        if list1[i1] <= list2[i2]:
            merged_list.append(list1[i1])
i1 += 1
        else:
            merged_list.append(list2[i2])
i2 += 1

    # One list is done, move what's left
    while i1 < len(list1):
        merged_list.append(list1[i1])
i1 += 1
    while i2 < len(list2):
        merged_list.append(list2[i2])
i2 += 1

    return merged_list
```

# merge sort recursive
def sort_r(list):
    if len(list) > 1:
        half_point = len(list)/2
        first_half = list[:half_point]
        second_half = list[half_point:]

        first_half_sorted = sort_r(first_half)
        second_half_sorted = sort_r(second_half)

        sorted_list = merge(first_half_sorted, second_half_sorted)
        return sorted_list
    else:
        return list
```

---

List of size 1. Base case
Recursion vs. Iteration

- There are usually similarities between an iterative solutions (e.g., looping) and a recursive solution.
  - In fact, anything that can be done with a loop can be done with a simple recursive function!
  - In many cases, a recursive solution can be easily converted into an iterative solution using a loop (but not always).

- Recursion can be very costly!
  - Calling a function entails overhead
  - Overhead can be high when function calls are numerous (stack overflow)
Recursion - the take home message

- Recursion is a great tool to have in your problem-solving toolbox.

- In many cases, recursion provides a natural and elegant solution to complex problems.

- If the recursive version and the loop version are similar, prefer the loop version to avoid overhead.

- Yet, even in these cases, recursion offers a creative way to *think* about how a problem could be solved.
Sample problem #1

- Write a function that calculates the sum of the elements in a list using a recursion

  *Hint: your code should not include ANY for-loop or while-loop!*

- Put your function in a module, import it into another code file and use it to sum the elements of some list.
Solution #1

```python
# utils.py

def sum_recursive(a_list):
    if len(a_list) == 1:
        return a_list[0]
    else:
        return a_list[0] + sum_recursive(a_list[1:])

# my_prog.py

my_list = [1, 3, 5, 7, 9, 11]

from utils import sum_recursive
print(sum_recursive(my_list))
```
Sample problem #2

- Write a **recursive** function that determines whether a string is a palindrome. Again, make sure your code does not include any loops.

A palindrome is a word or a sequence that can be read the same way in either direction.

For example:
- “detartrated”
- “olson in oslo”
- “step on no pets”
def is_palindrome(word):
    l = len(word)
    if l <= 1:
        return True
    else:
        return word[0] == word[l-1] and is_palindrome(word[1:l-1])

>>> is_palindrome("step on no pets")
True
>>> is_palindrome("step on no dogs")
False
>>> is_palindrome("12345678987654321")
True
>>> is_palindrome("1234")
False
Challenge problems

1. Write a recursive function that prime factorize s an integer number.

(The prime factors of an integer are the prime numbers that divide the integer exactly, without leaving a remainder).
Your function should print the list of prime factors:

```python
>>> prime_factorize(5624)
2 2 2 19 37
>>> prime_factorize(277147332)
2 2 3 3 3 3 7 7 11 23 23
```

Note: you can use a for loop to find a divisor of a number but the factorization process itself should be recursive!

2. Improve your function so that it “returns” a list containing the prime factors. Use pass-by-reference to return the list.

3. Can you do it without using ANY loops whatsoever?
import math

def prime_factorize(number):
    # find the first divisor
    divisor = number
    for i in range(2,int(math.sqrt(number))+1):
        if number % i == 0:
            divisor = i
            break
    print divisor,

    if divisor == number:  # number is prime. nothing more to do
        return
    else:                 # We found another divisor, continue
        prime_factorize(number/divisor)

prime_factorize(277147332)
import math

def prime_factorize(number, factors=[]):
    # find the first divisor
    divisor = number
    for i in range(2,int(math.sqrt(number))+1):
        if number % i == 0:
            divisor = i
            break

    factors.append(divisor)

    if divisor == number: # number is prime. nothing more to do
        return
    else:                 # We found another divisor, continue
        prime_factorize(number/divisor, factors)

factors = []
prime_factorize(277147332,factors)
print factors
Traversing a phylogenetic tree

- Recursion is extremely useful when processing a data structure that is recursive by nature.

```python
def preorder(node):
    if node == None:
        return

    print node.value,
    preorder(node.left)
    preorder(node.right)

preorder(root)
```

```
10 3 1 7 6 9 12 15
```
Traversing a phylogenetic tree

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```python
def preorder(node):
    if node == None:
        return

    print node.value,
    preorder(node.left)
    preorder(node.right)

preorder(root)
10 3 1 7 6 9 12 15
```

```python
def postorder(node):
    if node == None:
        return

    postorder(node.left)
    postorder(node.right)
    print node.value,

postorder(root)
1 6 9 7 3 15 12 10
```
Recursion vs. Iteration

- Check the following two solutions for the Fibonacci sequence:

```python
# Returns the n-th Fibonacci number
def fib_loop(n):
    prev = 0
    curr = 1
    for i in range(n-2):
        curr, prev = curr+prev, curr
    return curr

# Returns the n-th Fibonacci number
def fib_rec(n):
    if n == 1:
        return 0
    elif n == 2:
        return 1
    else:
        return fib_rec(n-1)+fib_rec(n-2)
```

Which one would you prefer?